A journey to the world of Numbers

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Chapter 1

The integers

We construct the integers starting from the natural numbers $\mathbb{N}$. Since `lean` already has a type called $\mathbb{Z}$, we define a new type called MyInt that will be another definition of the integers.

1.1 The *preintegers*

MyInt will be a quotient of a type called MyPreint.

Definition 1. Let MyPreint be $\mathbb{N} \times \mathbb{N}$.

Definition 2. We define a relation R on MyPreint as follows: (a, b) and (c, d) are related if and only if

$$a + d = b + c$$

Lemma 3. *R is a reflexive relation.*

Proof. This follows by commutativity of addition in $\mathbb{N}$. □

Lemma 4. *R is a symmetric relation.*

Proof. This follows by commutativity of addition in $\mathbb{N}$. □

Lemma 5. *R is a transitive relation.*

Proof. Let x, y and z in MyPreint such that xRy and yRz. We can write $x = (a, b)$ and similarly for $y = (c, d)$ and $z = (e, f)$. By assumption we have $a + d = b + c$ and $c + f = d + e$. Adding these equalities we get

$$a + d + c + f = b + c + d + e$$

Since addition on $\mathbb{N}$ is cancellative we get

$$a + f = b + e$$

as wanted. □

Lemma 6. *We have that R is an equivalence relation. From now on, we will write $x \approx y$ for xRy.*

Proof. Clear from Lemma 3, Lemma 4 and Lemma 5. □

Definition 7. We define an operation, called *negation* on MyPreint as follows: the negation of $x = (a, b)$ is (b, a):

$$-x = -(a, b) = (b, a)$$

Lemma 8. *If $x \approx x'$, then $-x \approx -x'$.*

Proof. Let $x = (a, b)$ and $x' = (a', b')$, so by assumption $a + b' = b + a'$. By definition we have

$$-x = -(a, b) = (b, a) \text{ and } -x' = -(a', b') = (b', a')$$

We need to show $b + a' = b' + a$, which follows immediately from $a + b' = b + a'$. □

Definition 9. We define an operation, called *addition* on MyPreint as follows: the addition of $x = (a, b)$ and $y = (b, c)$ is

$$x + y = (a, b) + (c, d) = (a + c, b + d)$$

Lemma 10. *If $x \approx x'$ and $y \approx y'$, then $x + y \approx x' + y'$.*

Proof. Let $x = (a, b)$, $y = (c, d)$, $x' = (a', b')$ and $y' = (c', d')$ such that $x \approx x'$ and $y \approx y'$. by assumption we have

$$a + b' = b + a' \text{ and } c + d' = d + c'$$

Adding these two equalities we get

$$a + b' + c + d' = b + a' + d + c'$$

and hence

$$a + c + b' + d' = b + d + a' + c'$$

that is $x + y = (a + c, b + d) \approx (a' + c', b' + d') = x' + y'$. □

Definition 11. We define an operation, called *multiplication* on MyPreint as follows: the multiplication of $x = (a, b)$ and $y = (b, c)$ is

$$x * y = (a, b) * (c, d) = (a * c + b * d, a * d + b * c)$$

Lemma 12. *If $x \approx x'$ and $y \approx y'$, then $x * y \approx x' * y'$.*

Proof. Let $x = (a, b)$, $y = (c, d)$, $x' = (a', b')$ and $y' = (c', d')$ such that $x \approx x'$ and $y \approx y'$. by assumption we have

$$a + b' = b + a' \text{ and } c + d' = d + c'$$

Multiplying the first equality by c' and by d' we get

$$a * c' + b' * c' = b * c' + a' * c' \tag{1.1}$$

and

$$b * d' + a' * d' = a * d' + b' * d' \tag{1.2}$$

Multiplying the second equality by a and by b we get

$$a * c + a * d' = a * d + a * c' \tag{1.3}$$

and

$$b * d + b * c' = b * c + b * d' \tag{1.4}$$

Adding (1.1) and (1.4) we get

$$a * c' + b' * c' + b * d + b * c' = b * c' + a' * c' + b * c + b * d'$$

Adding (1.3) and (1.2) we get

$$a * c + a * d' + b * d' + a' * d' = a * d + a * c' + a * d' + b' * d'$$

Adding the last two equations and cancelling the terms appearing on both sides we finally have

$$b' * c' + b * d + a * c + a' * d' = a' * c' + b * c + a * d + b' * d'$$

that is $x * y \approx x' * y'$. □

1.2 The integers

1.2.1 Definitions

Definition 13. We define our integers MyInt as

$$\text{MyInt} = \text{MyPreint} / \approx$$

We will write $[\![(a, b)]\!]$ for the class of (a, b).

Definition 14. We define the zero of MyInt, denoted 0 as the class of $(0, 0)$.

Definition 15. We define the one of MyInt, denoted 1 as the class of $(1, 0)$.

1.2.2 Commutative ring structure

Definition 16. We define the negation of $x = [\![(a, b)]\!]$ in MyInt as

$$-x = [\![-(a, b)]\!]$$

Thanks to Lemma 8 this is well defined.

Definition 17. We define the addition of $x = [\![(a, b)]\!]$ and $y = [\![(c, d)]\!]$ in MyInt as

$$x + y = [\![(a, b) + (c, d)]\!]$$

Thanks to Lemma 10 this is well defined.

Definition 18. We define the multiplication of $x = [\![(a, b)]\!]$ and $y = [\![(c, d)]\!]$ in MyInt as

$$x * y = [\![(a, b) * (c, d)]\!]$$

Thanks to Lemma 12 this is well defined.

Lemma 19. *Addition on* MyInt *is associative.*

Proof. To prove the lemma it is enough to prove that, for all a, b, c, d, e and f in $\mathbb{N}$, we have

$$([\![(a, b)]\!] + [\![(c, d)]\!]) + [\![(e, f)]\!] = [\![(a, b)]\!] + ([\![(c, d)]\!] + [\![(e, f)]\!])$$

Unravelling the definitions this is

$$a + c + e + (b + (d + f)) = b + d + f + (a + (c + e))$$

that is true by associativity and commutativity in $\mathbb{N}$. □

Proposition 20. MyInt *with addition and multiplication is a commutative ring.*

Proof. We have to prove various properties, namely:

- addition is associative (already done in Lemma 19)
- 0 works as neutral element for addition (on both sides)
- existence of an inverse for addition (we prove that $x + (-x) = (-x) + x = 0$)
- addition is commutative
- left and right distributivity of multiplication with respect to addition
- associativity of multiplication
- 1 works as neutral element for multiplication (on both sides)

All the proofs are essentially identical to the proof of Lemma 19 above. □

Lemma 21. *In* MyInt *we have* $0 \neq 1$.

Proof. If $0 = 1$ by definition we would have $[\![(0,0)]\!] = [\![(1,0)]\!]$ so $0 + 1 = 0 + 0$ in $\mathbb{N}$, that is absurd. □

Lemma 22. *Let* x *and* y *in* MyInt *such that* $x \neq 0$ *and* $y \neq 0$. *Then* $x * y \neq 0$.

Proof. It is enough to prove that, for all a, b, c and d in $\mathbb{N}$ such that $a \neq b$ and $c \neq d$ we have $a * c + b * d \neq a * d + b * c$. If this is not the case we must have $a = b$ or $c = d$. (Can you see how to prove this in $\mathbb{N}$? You cannot use subtraction!) and we are done. □

Lemma 23. *Let* x, y *and* z *in* MyInt *such that* $x \neq 0$ *and* $y * x = z * x$. *Then* $y = z$.

Proof. We have $0 = y * x - z * x = (y - z) * x$. Since $x \neq 0$ we have by Lemma 22 that $y - z = 0$ and hence $y = z$. □

1.2.3 The inclusion $i \colon \mathbb{N} \to \text{MyInt}$

Definition 24. We define a map

$$\begin{aligned} i \colon \mathbb{N} &\to \text{MyInt} \\ n &\mapsto [\![(n,0)]\!] \end{aligned}$$

Lemma 25. *We have that* $i(0) = 0$.

Proof. Clear from the definition. □

Lemma 26. *We have that* $i(1) = 1$.

Proof. Clear from the definition. □

Lemma 27. *For all* a *and* b *in* $\mathbb{N}$ *we have that*

$$i(a + b) = i(a) + i(b)$$

Proof. We have $i(a+b) = [\![(a+b,0)]\!]$, $i(a) = [\![(a,0)]\!]$ and $i(b) = [\![(b,0)]\!]$, so we need to prove that

$$[\![(a+b,0)]\!] = [\![(a,0)]\!] + [\![(b,0)]\!]$$

that is obvious from the definition. □

Lemma 28. *For all a and b in $\mathbb{N}$ we have that*

$$i(a*b) = i(a)*i(b)$$

Proof. We have $i(a*b) = [\![(a*b,0)]\!]$, $i(a) = [\![(a,0)]\!]$ and $i(b) = [\![(b,0)]\!]$, so we need to prove that

$$[\![(a*b,0)]\!] = [\![(a,0)]\!] * [\![(b,0)]\!]$$

By definition of multiplication we have

$$[\![(a,0)]\!] * [\![(b,0)]\!] = [\![(a*c+0*0, a*0+0*b)]\!]$$

and the lemma follows. □

Lemma 29. *We have that i is injective.*

Proof. Let a and b such that $i(a) = i(b)$. This means $[\![(a,0)]\!] = [\![(b,0)]\!]$ so $a+0 = 0+b$ and hence $a = b$. □

1.2.4 The order

Definition 30. Let x and y in MyInt. We write $x \leq y$ if there exist a natural number n such that

$$y = x + i(n)$$

Lemma 31. *The relation $\leq$ on* MyInt *is reflexive.*

Proof. We can just take $n = 0$. □

Lemma 32. *The relation $\leq$ on* MyInt *is transitive.*

Proof. Let x, y and z such that $x \leq y$ and $y \leq z$. It follows that there exist p and q such that $y = x + i(p)$ and $z = y + i(q)$. One can now take $p + q$ to show that $x \leq z$. □

Lemma 33. *The relation $\leq$ on* MyInt *is antisymmetric.*

Proof. Let x and y such that $x \leq y$ and $y \leq x$. It follows that there exist p and q such that $y = x + i(p)$ and $x = y + i(q)$. In particular

$$x = x + i(p) + i(q)$$

Since MyInt is a ring, we obtain $i(p) + i(q) = 0$. Moreover $i(p+q) = i(p) + i(q)$ and $i(0) = 0$, so $i(p+q) = i(0)$ and hence, since i is injective, $p + q = 0$. Now, p and q are *natural numbers*, so $p = q = 0$ and so $x = y$. □

It follows that $\leq$ is an order relation.

Lemma 34. *The order $\leq$ on* MyInt *is a total order.*

Proof. Let x and y by in MyInt. We can write $x = [\![(a,b)]\!]$ and $y = [\![(c,d)]\!]$ and we need to prove that $[\![(a,b)]\!] \leq [\![(c,d)]\!]$ or $[\![(c,d)]\!] \leq [\![(a,b)]\!]$. Let's consider two cases (we use that the order on $\mathbb{N}$ is total):

- if $a+d \leq b+c$ let e be such that $b+c = a+d+e$. We prove that $[\![(a,b)]\!] \leq [\![(c,d)]\!]$ using e. We have

$$[\![(a,b)]\!] + i(e) = [\![(a,b)]\!] + [\![(e,0)]\!] = [\![(a+e,b+0)]\!] = [\![(a+e,b)]\!]$$

  We have that $[\![(a+e,b)]\!] = [\![(c,d)]\!]$ since

$$a+e+d = b+d$$

  by our assumption on e and so $[\![(a,b)]\!] \leq [\![(c,d)]\!]$.

- the case $b+c \leq a+d$ is completely analogous.

□

Lemma 35. *We have that* MyInt *with* $\leq$ *is a* linear order

Proof. Clear. □

Lemma 36. *In* MyInt *we have that* $0 \leq 1$.

Proof. We use 1 (as natural number). We need to prove that $0 + i(1) = 1$. Unravelling the definitions this is obvious. □

Lemma 37. *Given two natural numbers a and b, we have $i(a) \leq b$ if and only if $a \leq b$.*

Proof.

- If $i(a) \leq i(b)$, let n be such that $i(b) = i(a) + i(n) = i(a+n)$. We obtain $b = a+n$ by injectivity of i and so $a \leq b$.

- If $a \leq b$, let k be such that $b = a+k$. We can use k to show that $i(a) \leq i(b)$.

□

1.2.5 Interaction between the order and the ring structure

Lemma 38. *Let x, y and z in* MyInt *be such that $x \leq y$. Then $z + x \leq z + y$.*

Proof. Let n be such that $y = x + i(n)$. It's immediate that n also work to show that $z + x \leq z + y$. □

Lemma 39. *Let x and y in* MyInt *be such that $0 < x$ and $0 < y$. Then $0 < x * y$.*

Proof. By Lemma 22 we already know that $x * y \neq 0$, so it is enough to prove that $0 \leq x * y$. Since $0 < x$, we have in particular that $0 \leq x$, and let n be such that $x = 0 + i(n)$. Similarly, let m be such that $y = 0 + i(m)$. We have $0 + i(n) = i(n)$ and $0 + i(m) = i(m)$, so we need to prove that $0 \leq i(n) * i(m)$. We do so using $n * m$: we have

$$0 + i(n * m) = i(n * m) = i(n) * i(m)$$

as required. □

Chapter 2

The rationals

We can now define the rationals, starting with our copy of the integers MyInt.

We follow a similar path to the one for MyInt.

2.1 The *prerationals*

MyRat will be a quotient of a type called MyPrerat.

Definition 40. Let MyPrerat be $\text{MyInt} \times \text{MyInt} \backslash \{0\}$

Definition 41. We define a relation R on MyPrerat as follows: (a, b) and (c, d) are related if and only if

$$a * d = b * c$$

Lemma 42. *R is a reflexive relation.*

Proof. Exercice. □

Lemma 43. *R is a symmetric relation.*

Proof. Exercice. □

Lemma 44. *R is a transitive relation.*

Proof. Exercice. □

Lemma 45. *We have that R is an equivalence relation. From now on, we will write $x \approx y$ for xRy.*

Proof. Clear from Lemma 42, Lemma 43 and Lemma 44. □

Definition 46. We define an operation, called *negation* on MyPrerat as follows: the negation of $x = (a, b)$ is $(-a, b)$:

$$-x = -(a, b) = (-a, b)$$

Note that it is automatically well defined (meaning that second component of $(-a, b)$ is different from 0).

Lemma 47. *If $x \approx x'$, then $-x \approx -x'$.*

Proof. Exercice. □

Definition 48. We define an operation, called *addition* on MyPrerat as follows: the addition of $x = (a, b)$ and $y = (b, c)$ is

$$x + y = (a, b) + (c, d) = (a * d + b * c, b * d)$$

Do you see why it is well defined?

Lemma 49. *If $x \approx x'$ and $y \approx y'$, then $x + y \approx x' + y'$.*

Proof. Exercice. □

Definition 50. We define an operation, called *multiplication* on MyPrerat as follows: the multiplication of $x = (a, b)$ and $y = (b, c)$ is

$$x * y = (a, b) * (c, d) = (a * c, b * d)$$

Lemma 51. *If $x \approx x'$ and $y \approx y'$, then $x * y \approx x' * y'$.*

Proof. Exercice. □

Definition 52. We define an operation, called *negation* on MyPrerat as follows: the inverse of $x = (a, b)$ is:

$$\text{if} b \neq 0 \text{ then } x^{-1} = (b, a), \text{ otherwise } x^{-1} = (0, 1)$$

Note that x^{-1} is *always* defined!

Lemma 53. *If $x \approx x'$, then $x^{-1} \approx x'^{-1}$.*

Proof. Exercice. □

2.2 The rationals

2.2.1 Definitions

Definition 54. We define our rationals MyRat as

$$\text{MyRat} = \text{MyPrerat} / \approx$$

We will write $[\![(a, b)]\!]$ for the class of (a, b).

Definition 55. We define the zero of MyRat, denoted 0 as the class of $(0, 1)$ (note that $1 \neq 0$ in MyInt).

Definition 56. We define the one of MyRat, denoted 1 as the class of $(1, 1)$ (note that $1 \neq 0$ in MyInt).

2.2.2 Commutative ring structure

Definition 57. We define the negation of $x = [\![(a,b)]\!]$ in MyInt as

$$-x = [\![-(a,b)]\!]$$

Thanks to Lemma 47 this is well defined.

Definition 58. We define the addition of $x = [\![(a,b)]\!]$ and $y = [\![(c,d)]\!]$ in MyInt as

$$x + y = [\![(a,b) + (c,d)]\!]$$

Thanks to Lemma 49 this is well defined.

Definition 59. We define the multiplication of $x = [\![(a,b)]\!]$ and $y = [\![(c,d)]\!]$ in MyInt as

$$x * y = [\![(a,b) * (c,d)]\!]$$

Thanks to Lemma 51 this is well defined.

Definition 60. We define the negation of $x = [\![(a,b)]\!]$ in MyInt as

$$x^{-1} = [\![(a,b)^{-1}]\!]$$

Thanks to Lemma 53 this is well defined.

Proposition 61. MyRat *with addition and multiplication is a commutative ring.*

Proof. We have to prove various properties, namely:

- addition is associative
- 0 works as neutral element for addition (on both sides)
- existence of an inverse for addition (we prove that $x + (-x) = (-x) + x = 0$)
- addition is commutative
- left and right distributivity of multiplication with respect to addition
- associativity of multiplication
- 1 works as neutral element for multiplication (on both sides)

All the proofs are essentially identical, going to MyInt, unravelling the definition and then checking the equality holds in MyInt. □

Lemma 62. *In* MyRat *we have* $0 \neq 1$.

Proof. If $0 = 1$ by definition we would have $[\![(0,1)]\!] = [\![(1,1)]\!]$ so $0 * 1 = 1 * 0$ in MyInt, that is absurd. □

Lemma 63. *Let* $x \neq 0$ *be in* MyRat. *Then* $x * x^{-1} = 1$.

Proof. Let $x = [\![(a,b)]\!]$, with $b \neq 0$. Since $x \neq 0$ we have $a \neq 0$ and so $x^{-1} = [\![(b,a)]\!]$. The lemma follows by definition of multiplication. □

Proposition 64. MyRat *with addition and multiplication is a field.*

Proof. Clear because of Lemma 63. □

2.2.3 The inclusion $i\colon \mathbb{N} \to \mathrm{MyRat}$

Definition 65. We define a map

$$\begin{aligned} i\colon \mathbb{N} &\to \mathrm{MyRat} \\ n &\mapsto [\![(\mathrm{MyInt}.in, 1)]\!] \end{aligned}$$

Lemma 66. *We have that $i(0) = 0$.*

Proof. Clear from the definition. □

Lemma 67. *We have that $i(1) = 1$.*

Proof. Clear from the definition. □

Lemma 68. *For all a and b in $\mathbb{N}$ we have that*

$$i(a + b) = i(a) + i(b)$$

Proof. We have $i(a + b) = [\![(\mathrm{MyInt}.i(a + b), 1)]\!] = [\![(\mathrm{MyInt}.i(a), 1) + (\mathrm{MyInt}.i(b), 1)]\!]$, $i(a) = [\![(\mathrm{MyInt}.i(a), 1)]\!]$ and $i(b) = [\![(\mathrm{MyInt}.i(b), 1)]\!]$, so we need to prove that

$$[\![(\mathrm{MyInt}.i(a), 1) + (\mathrm{MyInt}.i(b), 1)]\!] = [\![(\mathrm{MyInt}.i(a), 1)]\!] + [\![(\mathrm{MyInt}.i(b), 1)]\!]$$

that is obvious from the definition (). □

Lemma 69. *For all a and b in $\mathbb{N}$ we have that*

$$i(a * b) = i(a) * i(b)$$

Proof. Similar to the proof of Lemma 68. □

Lemma 70. *We have that i is injective.*

Proof. Let a be such that $i(a) = 0$. This means $[\![(\mathrm{MyInt}.ia, 1)]\!] = [\![(0, 1)]\!]$ so $(\mathrm{MyInt}.ia) * 1 = 1 * 0$ and hence $\mathrm{MyInt}.ia = 0$ so $a = 0$. □

2.2.4 The inclusion $j\colon \mathrm{MyInt} \to \mathrm{MyRat}$

Definition 71. We define a map

$$\begin{aligned} j\colon \mathrm{MyInt} &\to \mathrm{MyRat} \\ n &\mapsto [\![(n, 1)]\!] \end{aligned}$$

Lemma 72. *We have that $j(0) = 0$.*

Proof. Clear from the definition. □

Lemma 73. *We have that $j(1) = 1$.*

Proof. Clear from the definition. □

Lemma 74. *For all a and b in* MyInt *we have that*

$$j(a + b) = j(a) + j(b)$$

Proof. Exercice. ☐

Lemma 75. *For all a and b in $\mathbb{N}$ we have that*

$$j(a * b) = j(a) * j(b)$$

Proof. Exercice. ☐

Lemma 76. *We have that j is injective.*

Proof. Exercice. ☐

Lemma 77. *Let n be a natural number. Then* $\mathrm{MyRat}.j(\mathrm{MyInt}.i(n)) = MyRat.i(n)$.

Proof. It follows from unravelling all the definitions. ☐

Lemma 78. *Let a and b be in* MyInt *with $b \neq 0$. Then* $[\![(a, b)]\!] = j(a) * j(b)^{-1}$.

Proof. Exercice. ☐

2.2.5 Nonegativity

Before defining the order on MyRat, let's define the notion of nonnegativity.

Definition 79. Given $x = (a, b)$ in MyRat, we say that x is *nonnegative* if $0 \leq a$ and $0 < b$.
Can you see why it corresponds to the "usual definition" when we think that $x = a/b$?

Lemma 80. *We have that* 0 *in* MyRat *is nonnegative.*

Proof. Obvious. ☐

Lemma 81. *We have that* 1 *in* MyRat *is nonnegative.*

Proof. Obvious. ☐

Lemma 82. *Let x be in* MyRat *such that both x and $-x$ are nonnegative. Then $x = 0$.*

Proof. Unravelling all the definitions we end up with a, b, c and d in MyInt such that $0 \leq a$, $0 < b$, $0 \leq c$, $0 < d$ and $-(a * d) = b * c$. This implies $a = 0$. ☐

Lemma 83. *Let x be in* MyRat *such that x is not nonnegative. Then $-x$ is nonnegative.*

Proof. Annoying but easy, left as an exercice. ☐

Lemma 84. *Let x and y be in* MyRat *both nonnegative. Then $x + y$ is nonnegative.*

Proof. Exercice. ☐

Lemma 85. *Let x and y be in* MyRat *both nonnegative. Then $x * y$ is nonnegative.*

Proof. Exercice. ☐

Lemma 86. *Let x be in* MyRat *be nonnegative. Then x^{-1} is nonnegative.*

Proof. Exercice. ☐

2.2.6 The order

Definition 87. Let x and y in MyRat. We write $x \leq y$ if $y - x$ is nonnegative.

Lemma 88. *We have that* $0 \leq x$ *if and only if* x *is nonnegative.*

Proof. Clear. □

Lemma 89. *In* MyRat *we have that* $0 \leq 1$.

Proof. Clear because of Lemma 81. □

Lemma 90. *The relation* $\leq$ *on* MyRat *is reflexive.*

Proof. Clear because of Lemma 80. □

Lemma 91. *The relation* $\leq$ *on* MyRat *is transitive.*

Proof. It follows from Lemma 84. □

Lemma 92. *The relation* $\leq$ *on* MyRat *is antisymmetric.*

Proof. It follows from Lemma 82. □

It follows that $\leq$ is an order relation.

2.2.7 Interaction between the order and the ring structure

Lemma 93. *Let* x*,* y *and* z *in* MyRat *be such that* $x \leq y$*. Then* $z + x \leq z + y$.

Proof. Clear from the definitions. □

Lemma 94. *Let* x *and* y *in* MyRat *be such that* $0 \leq x$ *and* $0 \leq y$*. Then* $0 \leq x * y$.

Proof. It follows from Lemma 85. □

We have proved that MyRat is an ordered ring.

2.2.8 Interaction between the order and the inclusions

Lemma 95. *Let* x *and* y *in* MyInt*. We have that* $j(x) \leq j(y)$ *if and only if* $x \leq y$.

Proof. Exercice. □

Lemma 96. *Let* x *and* y *in* $\mathbb{N}$*. We have that* $i(x) \leq i(y)$ *if and only if* $x \leq y$.

Proof. It follows immediately by Lemma 37, Lemma 95 and Lemma 77. □

2.2.9 The linear order structure

Lemma 97. *The order* $\leq$ *on* MyRat *is a total order.*

Proof. This follows by Lemma 83. ☐

Lemma 98. *We have that* MyRat *with* $\leq$ *is a* linear order

Proof. Clear from the lemma above. ☐

Lemma 99. *Let* x *and* y *in* MyRat *be such that* $0 < x$ *and* $0 < y$*. Then* $0 < x * y$*.*

Proof. Exercice. ☐

We now have that MyRat is a *strict ordered ring*: a nontrivial ring with a partial order such that addition is strictly monotone and multiplication by a positive number is strictly monotone.